6. The Lin-Kernighan Heuristic for the Symmetric TSP

15.01.12

We consider the symmetric TSP on a graph $G=(V,E)$ with edge weights $\|e\| = c_e$.

Def. 6.1 (k-opt move): Let $T \subseteq E$ be a tour in G, $k \in \mathbb{N}$,

$X = \{x_1, \dots, x_k\} \subseteq T$ set of out-edges, $X_i := \{x_1, \dots, x_i\}$, $i = 1, \dots, k$

$Y = \{y_1, \dots, y_k\} \subseteq E \setminus T$ set of in-edges, $Y_i := \{y_1, \dots, y_i\}$, $i = 1, \dots, k$

$X \to Y$ k-opt move for T, if $T \setminus X \cup Y$ is a tour

$G(X,Y) := \|X\| - \|Y\|$ level / gain of k-opt move $X \to Y$ (w.r.t. T), $G_i := G(X_i, Y_i)$, $i = 1, \dots, k$

$X \to Y$ improving $:\Longleftrightarrow$ $G(X,Y) > 0$

T k-optimal $:\Longleftrightarrow$ $\nexists$ improving k-opt move for T.

Idea: Construct X, Y element by element, i.e.,

$X_1 = \{x_1\} \to \{y_1\} = Y_1$, $\quad g_1 = \|x_1\| - \|y_1\|$, $G_1 = g_1 > 0$

$X_2 = \{x_1, x_2\} \to \{y_1, y_2\} = Y_2$, $\quad g_2 = \|x_2\| - \|y_2\|$, $G_2 = g_1 + g_2 > 0$

$X_k = \{x_1, \dots, x_k\} \to \{y_1, \dots, y_k\} = Y_k$, $\quad g_k = \|x_k\| - \|y_k\|$, $G_k = \sum_{i=1}^{k} g_i > 0$.

Allow g_i negative, as long as $G(X_k, Y_k)$ is positive to escape local optima.

Lemma 6.2 (gasoline lemma, Lin & Kernighan [1973]): Let $g_0, \dots, g_{n-1} \in \mathbb{R}$ be a sequence of n real numbers s.t. $\sum_{i=0}^{n-1} g_i > 0$. Then there is an index $k \in \{0, \dots, n-1\}$ such that the partial sums

$$\sum_{i=k}^{k+j} g_{i \bmod n} > 0, \quad j = 0, \dots, n-1.$$

Proof: Let k be the largest index s.t. $g_0 + \dots + g_{k-1}$ is minimal. Then

$$g_k + \dots + g_j = (g_0 + \dots + g_j) - (g_0 + \dots + g_{k-1}) > 0, \quad k \le j \le n-1,$$

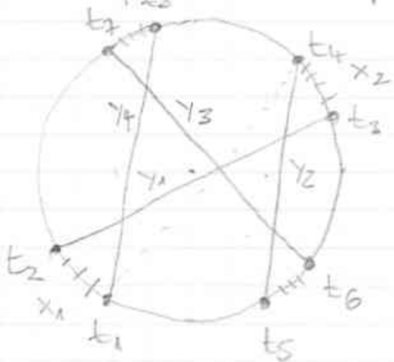
$$g_k + \dots + g_{n-1} + g_0 + \dots + g_j \ge g_k + \dots + g_{n-1} + g_0 + \dots + g_{k-1} > 0, \quad 0 \le j < k \quad \square$$

Problem 6.3 (Lovász [1979, Problem 3.21]): Along a speed track there are some gas-stations. The total amount of gasoline available in them is equal to what our car (which has a very large tank) needs for going around the track. Prove that there is a gas-station such that if we start there with an empty tank, we shall be able to go around the track without running out of gasoline.

Def. 6.4 (Sequential k-opt move): A k-opt move $X \to Y$ is a sequential exchange k-opt

$$X = \{x_1, \dots, x_k\} = \{t_{2i-1} t_{2i}\}_{i=1}^{k}$$
$$Y = \{y_1, \dots, y_k\} = \{t_{2i} t_{2i+1}\}_{i=1}^{k}$$
$$t_1 = t_{2k+1}.$$

Ex. 6.5 (Sequential 4-opt move):



Def. 6.6: A sequential exchange k-opt $X \to Y$ is

i) disjoint $:\Leftrightarrow X \cap Y = \emptyset$

ii) improving $:\Leftrightarrow G_i = G(\underbrace{X_i}_{\ni x_i}, Y_i) = \sum_{i=1}^{k} g_i > 0, \ i = 1, \dots, k$

iii) feasible $:\Leftrightarrow \{t_1 t_2, \dots, t_{2i-1} t_{2i}\} \to \{t_2 t_3, \dots, t_{2i} t_1\}$ is an i-opt move, $i = 2, \dots, k$.

iv) restricted $:\Leftrightarrow t_{2i+1} \in N_i(t_{2i})$ for some (small) neighborhood of t_{2i}, $i = 1, \dots, k$, depending on the level i.

v) sequential $:\Leftrightarrow X \to Y$ is disjoint, improving, feasible.

Obs. 6.7: i) There is a 1-1 correspondence between sequential k-opt moves and negative alternating cycles of length 2k in G. Note that negative alternating cycles can be found in polynomial time.

ii) Consider $X_i \to Y_i$, $i = 2, \dots, k$. If

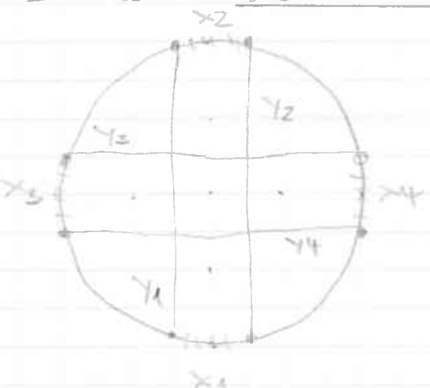
$$X_i = \{\underbrace{t_1 t_2}_{= x_1}, \dots, \underbrace{t_{2i-3} t_{2(i-1)}}_{= x_{i-1}}\} \cup \{\underbrace{t_{2i-1} t_{2i}}_{= x_i}\}$$
$$\to Y_i = \{\underbrace{t_2 t_3}_{= y_1}, \dots, \underbrace{t_{2i-2} t_{2i-1}}_{= y_{i-1}}\} \cup \{\underbrace{t_{2i} t_1}_{\neq y_i,\ i = 2, \dots, k-1}\}$$

is an i-opt move, there is no choice for t_{2i} and hence for x_i, i.e., the degrees of freedom in a sequential k-opt move are the choices of $x_1, y_1, \dots, y_k$, while $x_2, \dots, x_k$ are uniquely determined.

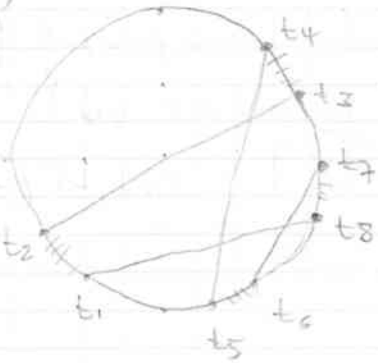
iv) every improving 2- and 3-opt move is sequential. (→ Ex.)

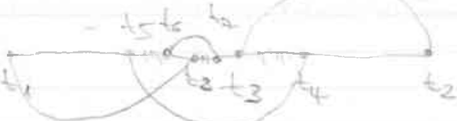
v) There are 4-opt moves that are not sequential, e.g., the

So-called <u>double-bridge move</u> (Martin, Otto & Felten [1992]) (→ Ex).



iii) $X_i \to Y_i$ produces a 1-tree, $i=1,\dots,k-1$





This gives rise to a data structure in which X_i, $i=2,\dots,k$ is apparent (Johnson & McGeoch [1995]).

vi) Popular restrictions are

α) Nearest neighbors (Lin & Kernighan [1973]):

$N_1(v) = N_2(v) = \text{argmin}_5\, c_{vw}$, $N_i(v) = 1$, $i \geq 3$

β) Nearest neighbors (Helsgaun [2006]):

$N_i(v) = \{ w \in V :$ Voronoi regions of v and w are neighboring $\}$

α-Tolerances (Helsgaun [2006]):

γ) Let $1 \in V$, T a min minimal 1-tree and for $vw \in E$ let $T(vw)$ be a minimal 1-tree containing edge e. Helsgaun [2006] observed $|C \cap T| \approx 0.7-0.8$ for an optimal tour C. Let

$$\alpha(vw) := c(T(vw)) - c(T) \quad \forall vw \in E \qquad \underline{\alpha\text{-tolerance of edge } e}$$

$$N_1(v) = N_2(v) = \text{argmin}_5\, \alpha(vw), \quad N_i(v) = \text{argmin}\, \alpha(vw),\ i \geq 3$$

a) $\alpha(vw)$ can be computed by adding vw to T and deleting the most expensive edge in the resulting cycle (→ Ex) in $O(n^2)$ time.

b) $\alpha(vw) = 0$ if $vw \in T$ (→ Ex.)

c) α can be computed in $O(n^2)$ time

ε) β-Tolerances (Helsgaun [2006]):

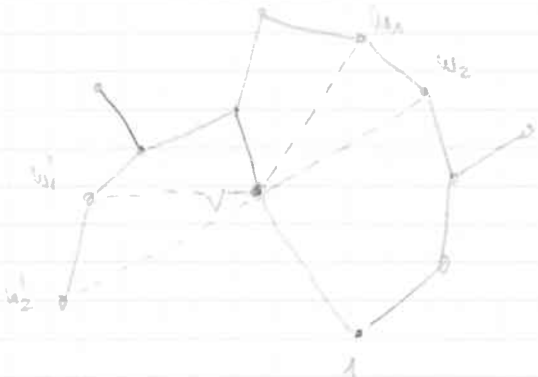
$$\beta(vw) := c(vw) - \alpha(vw)$$

$$N_1(v) = N_2(v) = \text{argmin}_5\, \beta(vw), \quad N_i(v) = \text{argmin}\, \beta(vw),\ i \geq 3$$

β can be computed in $O(n^2)$ time (and N in $O(n)$ space).

Lemma 6.8: $p(v\cdot)$ can be computed in $O(n)$ time for every fixed $v \in V$.

Proof (Sketch): $p(vw_2)$ can be computed from $p(vw_1)$. □



Alg. 6.9: the Lin-Kernighan-heuristic (Lin & Kernighan [1973]):

Input: $G=(V,E)$, $|e| = c_e \ \forall e \in E$, tour T, $N_i(v) \subseteq p(v)$ sorted $\forall v \in V, i=1..\mu$

Output: Tour T^* s.t. $c(T^*) \leq c(T)$

1. $i \leftarrow 1$, $G \leftarrow 0$, $G^* \leftarrow 0$, $T^* \leftarrow T$.

   Choose $t_1 \in V$.

   Choose $t_2 \in V$ s.t. $x_1 = t_1 t_2 \in T$,

   Choose $t_3 \in V$ s.t. $y_1 = t_2 t_3 \notin T$ and $g_1 = G(x_1, y_1) > 0$.

2. $i \leftarrow i+1$

   Choose $t_{2i} \in V$ s.t. $X_{i-1} \cup \{t_{2i-1} t_{2i}\} \to Y_{i-1} \cup \{t_{2i} t_1\}$ is feasible and disjoint

   If $G + |t_{2i-1} t_{2i}| - |t_{2i} t_1| > G^*$ then

   $G^* \leftarrow G + |t_{2i-1} t_{2i}| - |t_{2i} t_1|$

   $T^* \leftarrow T \setminus (X_{i-1} \cup \{t_{2i-1} t_{2i}\}) \cup (Y_{i-1} \cup \{t_{2i} t_1\})$

3. Choose "next best" $t_{2i+1} \in N_i(t_{2i})$ s.t. $X_{i-1} \cup \{t_{2i-1} t_{2i}\} \to Y_{i-1} \cup \{t_{2i} t_{2i+1}\}$ is improving and disjoint

   $X_i \leftarrow X_{i-1} \cup \{t_{2i-1} t_{2i}\}$

   $Y_i \leftarrow Y_{i-1} \cup \{t_{2i} t_{2i+1}\}$

   $g_i \leftarrow |t_{2i-1} t_{2i}| - |t_{2i} t_{2i+1}|$

   $G \leftarrow G + g_i$

   goto 2.

4. If no such t_{2i+1} exists and $i \geq 3$ then

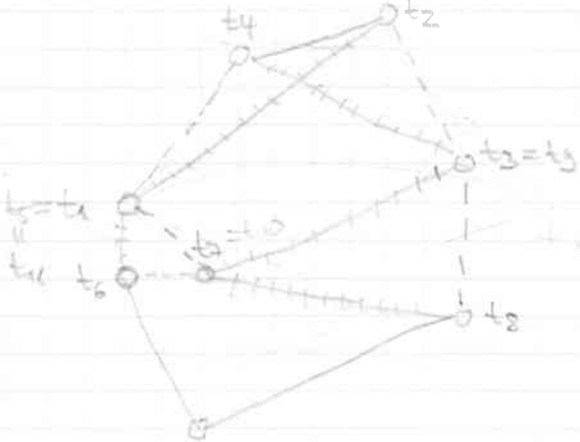
   $i \to i-1$, goto 3.

5. If $i = 2$ goto 1 and try the alternative t_2 ("alternate step").

6. If $i = 1$ goto 1 and try t_1 not yet used.

7. Output T^*.

Ex. 6.10 (Lin-Kernighan-Heuristic, Euclidean TSP (DSOR [2012]):



after 1 (or 2) LK-Searches

Optimal tour

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